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Asymptotic Stabilization of Nonlinear Systems with State Constraints

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ABSTRACT

This paper presents asymptotic stabilization strategy for nonlinear systems, in which the individual states are constrained within certain ranges. A barrier function whose value increases to infinity as the argument approaches the boundary of the constrained region is utilized as a Lyapunov function candidate. The derived control law guarantees that, for all initial conditions within the constraint range, the asymptotic stability of the origin is achieved and the state constraints are not violated for all time. The control gains assign the desired spectrum of the closed-loop system in a neighborhood of the origin, and a fast convergence of the states from initial conditions to the origin is achieved. By simulation, the performance of the proposed control law is compared with that of the control law derived upon a quadratic Lyapunov function and that of the nonlinear model predictive control.

Keywords: Barrier function, Lyapunov method, nonlinear systems, state constraints, simulation.

Mathematics Subject Classification: 93C10

Computing Classification System: G.1.7

1. INTRODUCTION

Nonlinear control problems with state constraints exist in many physical systems including robotics (Widyotriatmo, 2012; Zhang, Shang, & Gao, 2013), electrostatic microactuator (Tee, Ge, & Tay, 2009a), magnetic bearings systems (Do, 2010), and aeroelastic systems (Xiang, Liu, & Liu, 2013). Constraint violations can cause poor performance or component degradation, and damage of the system. Asymptotic stabilization of nonlinear systems without violating the constraints is of great importance in many applications. In this paper, a control design for uncertain nonlinear systems with state constraints is focused.

For linear systems, the set invariance method (Blanchini, 1999) can be used to handle state constraints. For nonlinear systems, the model predictive controls (MPCs) (Mayne et al., 2000;

Findeisen et al., 2003; Mhaskar, El-Farra, & Christofides, 2006; Bravo, Alamo, & Camacho, 2006) and the reference governor methods (Bemporad, 1998; Gilbert & Ong, 2011) have been proposed. The MPC utilizes an iterative finite horizon optimization of the plant model. The reference governor methods modify the reference signal using an optimization algorithm to avoid constraint violation. For successful implementation of the methods, a proper optimization problem should be well formulated. In (Mahindrakar & Sankaranarayanan, 2008; Ding, 2009) linear-matrix-inequality-based (LMI-based) optimization methods were utilized in the control design of nonlinear systems with state constraints. In (Kristic & Bement, 2006), a backstepping method with appropriate control gains was used to ensure a small overshoot of tracking response. In (Liu & Zhao, 2012), fuzzy systems are utilized in backstepping design procedure. Necessary conditions for optimality for regular solutions of the terminal optimal control problems with state and endpoint constraints are presented in (Imanov, 2012).

The Lyapunov method allows the direct design of a control law, which is embedded in an appropriately chosen Lyapunov function (Freeman & Kokotovic, 1996; Khalil, 2002; Widyotriatmo & Hong, 2011; Rehan & Hong, 2011). A quadratic function is usually utilized as a Lyapunov function candidate. Recently, for the systems with state constraints, the use of barrier functions was introduced: In (Tee & Ge, 2011), a barrier function method for output tracking of a state-constrained nonlinear system in strict feedback form was presented. However, in this work, an offline optimization method was utilized to determine the control parameters for maintaining the states inside the ranges of constraints. For the case of an asymptotic stabilization problem, it was shown that the initial conditions close to the boundary of a constraint region are not feasible to converge to the origin (Tee, Ge, & Tay, 2009b). On the other hand, for the systems with output constraints, the tracking control of a single-input-single-output (SISO) system using the barrier-function-based method was applied to electrostatic parallel plate microactuators (Tee, Ge, & Tay, 2009a) and to magnetic bearings (Do, 2010).

In this paper, a control law for a nonlinear system utilizing a barrier function as a Lyapunov function candidate is designed. Using the proposed control law, the time derivative of the Lyapunov function becomes negative semidefinite, and the convergence of the states from all the initial conditions within constraints to the origin is guaranteed. The boundedness of the barrier functions ensures that the states never violate their constraints.

Contributions of this paper are the following. First, a control law that achieves the asymptotic stability of the origin of a nonlinear system with state constraints is proposed. Second, using the proposed control law, all the initial conditions inside the constraints are forced to converge to the origin, while not violating the state constraints for all time. Thus, the method can be a viable solution to the problem in which any violation of constraints is absolutely prohibited. Third, the proposed method does not require an offline optimization method for determining the control parameters. Fourth, the proposed method can be applied to multi-input-multi-output MIMO systems. Fifth, the control gains determine the performance of the closed-loop system in a neighborhood of the origin, and a fast convergence of the states to the origin is achieved. Sixth, the effectiveness of the proposed method is shown by comparing the performance of the proposed control law with the one derived from a quadratic Lyapunov function, and that based on the nonlinear model predictive control (NMPC) in (Grune & Pannek, 2011).

This paper is organized as follows. Section 2 provides the problem formulation including the considered system and the properties of a barrier function. Section 3 discusses the derivation of control law based upon the barrier function as a Lyapunov function candidate. Section 4 presents simulation results of the proposed control law showing the effectiveness of the methodology, and the comparison of the proposed control law with other two methods available in the literature. Section 5 draws conclusions.

2. PROBLEM FORMULATION

Let $x_p(t) = [x_{p,1}(t), \ldots, x_{p,n_p}(t)]^T \in R^{n_p}$ and $x_q(t) = [x_{q,1}(t), \ldots, x_{q,n_q}(t)]^T \in R^{n_q}$ be the partitions of the state vector $x(t) = [x_p^T(t), x_q^T(t)]^T$, and $\theta_p(t) = [\theta_{p,1}(t), \ldots, \theta_{p,n_{\theta,p}}(t)]^T \in R^{n_{\theta,p}}$ and $\theta_q(t) = [\theta_{q,1}(t), \ldots, \theta_{q,n_{\theta,q}}(t)]^T \in R^{n_{\theta,q}}$ be the unknown parameter vectors, where $\theta(t) = [\theta_p^T(t), \theta_q^T(t)]^T$. The following nonlinear system is considered.

$$\dot{x}_p(t) = F_p(x(t))\theta_p(t) + G_p(x(t))x_q(t), \tag{1}$$

$$\dot{x}_q(t) = F_q(x(t))\theta_q(t) + G_q(x(t))u(t), \tag{2}$$

where $F_p(x(t))$: $R^{n_p+n_q} \to R^{n_p \times n_{\theta,p}}$, $G_p(x(t))$: $R^{n_p+n_q} \to R^{n_p \times n_q}$, $F_q(x(t))$: $R^{n_p+n_q} \to R^{n_q \times n_{\theta,q}}$, and $G_q(x(t))$: $R^{n_p+n_q} \to R^{n_q \times n_q}$ are smooth mappings, and $u(t) \in R^{n_q}$ is the input. Let $\bar{x}_{p,i}$ and $\bar{x}_{q,i}$ be the upper bounds of $x_{p,i}$ and $x_{q,i}$, respectively. The control objective is to achieve the asymptotic stability of the origin, from the intial values $x_p(0) \in \Omega_p = \{ x_p(t) \in R^{n_p} : |x_{p,i}(t)| < \bar{x}_{p,i}, i = 1, \ldots, n_p \}$, and $x_q(0) \in \Omega_q = \{ x_q(t) \in R^{n_q} : |x_{q,i}(t)| < \bar{x}_{q,i}, i = 1, \ldots, n_q \}$, while keeping $x_p(t) \in \Omega_p$ and $x_q(t) \in \Omega_q$ for all time. The following assumptions are made.

Assumption 1: $|\theta_{p,i}| \le \bar{\theta}_{p,i}$, $i = 1, \ldots, n_{\theta,p}$, and $|\theta_{q,i}| \le \bar{\theta}_{q,i}$, $i = 1, \ldots, n_{\theta,q}$, where $\bar{\theta}_{p,i}$ and $\bar{\theta}_{q,i}$ are the upper bounds of $\theta_{p,i}$ and $\theta_{q,i}$, respectively .

Assumption 2: $\left\|F_p(x(t))\right\| \le k_{F_p} \left\|x_q(t)\right\|$ and $\left\|F_q(x(t))\right\| \le k_{F_q} \left\|x_q(t)\right\|$ for all time, where k_{Fp} and k_{Fq} are positive constants.

Assumption 3: $G_q^{-1}(x(t))$ exists for all time.

Remark 1. Assumption 1 limits the upper bounds of the uncertainties. Assumption 2 indicates that the uncertain term of the system can be directly compensated by the control input. Assumption 3 ensures that the control input always exists for all time. The three assumptions are adopted from the literature that discusses nonlinear control problem (Freeman & Kokotovic, 1996 ; Khalil, 2002).

To prevent all the states of x from violating their constraints, the following barrier functions for individual states, $V_i(x_i(t))$: $(-\bar{x}_i, \bar{x}_i) \to R^+$, $i = 1, \ldots, n_p + n_q$, are utilized as a Lyapunov function (Ngo, Mahony, & Jiang, 2005; Tee, Ge, & Tay, 2009b):

$$V_i(x_i(t)) = \frac{1}{2}\ln\left(\frac{\bar{x}_i^2}{\bar{x}_i^2 - x_i^2(t)}\right), \tag{3}$$

which are continuous on $(-\bar{x}_i, \bar{x}_i)$, positive definite, and $V_i(x_i(t)) \to \infty$ as $x_i(t) \to \pm\bar{x}_i$. The following lemma shows the use of barrier functions as a Lyapunov function candidate.

Lemma. (Tee, Ge, Tay, 2009b; Do, 2010) Let $z(t) = [z_1(t), \ldots, \ z_{n_z}(t)]^T \in R^{n_z}$. For some constants $\bar{z}_i$, $i = 1, .., n_z$, let $\Omega_z = \{z(t) \in R^{n_z} : |z_i(t)| < \bar{z}_i, i = 1, \ldots, n_z\}$. Consider the system

$$\dot{z}(t) = f(t, z), \tag{4}$$

where f: $R^+ \times R^{n_z} \to R^{n_z}$. Let $V_i : (-\bar{z}_i, \bar{z}_i) \to \mathrm{R}^+$, $i = 1, \ldots, n_z$, be positive definite functions that are continuously differentiable on Ω_z. Let $V_i(z_i) \to \infty$ as $z_i \to \pm\bar{z}_i$, $i = 1, \ldots, n_z$. Let $V(z(t)) = \sum_{i=1}^{n_z} V_i(z_i(t))$ and $z(0) \in \Omega_z$. If

$$\dot{V}(z(t)) \le 0, \tag{5}$$

in the set Ω_z, it follows that $z(t) \in \Omega_z$ for $t \in [0, \infty)$.

Proof. Since V is a positive definite function, the negative semidefinite form in (5) implies that $V(z(t)) \le V(z(0)) \quad \forall t \in [0, \infty)$, that is, $V(z(t))$ is bounded $\forall t \in [0, \infty)$. Since $V(z) = \sum_{i=1}^{n_z} V_i(z_i)$ is a positive definite function, then $\sum_{i=1}^{n_z} V_i(z_i)$ becomes bounded $\forall t \in [0, \infty)$. From the property of the barrier function, that is, $V_i(z_i) \to \infty$ only if $z_i \to \pm\bar{z}_i$, $i = 1, \ldots, n_z$, and given that $z_i(0) \in \Omega_z$, it can be concluded that $z_i(t) \in \Omega_z$, $i = 1, \ldots, n_z$, $\forall t \in [0, \infty)$.☐

3. CONTROL DESIGN

Let a Lyapunov function candidate for (1)-(2) be introduced as

$$V = \frac{1}{2}\sum_{i=1}^{n_p} k_{p,i} \ln\left(\frac{\bar{x}_{p,i}^T \bar{x}_{p,i}}{\bar{x}_{p,i}^2 - x_{p,i}^2}\right) + \frac{1}{2}\sum_{i=1}^{n_q} \ln\left(\frac{\bar{x}_{q,i}^T \bar{x}_{q,i}}{\bar{x}_{q,i}^2 - x_{q,i}^2}\right), \tag{6}$$

where $k_{p,i}$, $i = 1, \ldots, n_p$, are positive constants. The time derivative of V is as follows

$$\begin{aligned} \dot{V} &= \dot{x}_p^T K_p \Phi_p^{-1} x_p + \dot{x}_q^T \Phi_q^{-1} x_q \\ &= \theta_p^T F_p^T K_p \Phi_p^{-1} x_p + x_q^T G_p^T K_p \Phi_p^{-1} x_p + \theta_q^T F_q^T \Phi_q^{-1} x_q + u^T G_q^T \Phi_q^{-1} x_q. \end{aligned} \tag{7}$$

where $K_p = \operatorname{diag}\left(k_{p,1}, \ldots, k_{p,n_p}\right)$, and Φ_p and Φ_q are as follows.

$$\Phi_p = \operatorname{diag}\left(\bar{x}_{p,1}^2 - x_{p,1}^2, \ldots, \bar{x}_{p,n_p}^2 - x_{p,n_p}^2\right), \tag{8}$$

$$\Phi_q = \operatorname{diag}\left(\bar{x}_{q,1}^2 - x_{q,1}^2, \ldots, \bar{x}_{q,n_q}^2 - x_{q,n_q}^2\right). \tag{9}$$

Let the control law be designed as

$$u = -G_q^{-1}\left(\Phi_q G_p^T K_p \Phi_p^{-1} x_p + K_q x_q + v\right), \tag{10}$$

where $K_q = \operatorname{diag}\left(k_{q,1}, \ldots, k_{q,n_q}\right)$ is a diagonal constant matrix, $k_{q,i}$, $i = 1, \ldots, n_q$, are positive constants, and v is to be designed. The time derivative of V becomes

$$\begin{aligned} \dot{V} &= -x_q^T K_q \Phi_q^{-1} x_q - v^T \Phi_q^{-1} x_q + \theta_p^T F_p^T K_p \Phi_p^{-1} x_p + \theta_q^T F_q^T \Phi_q^{-1} x_q \\ &\le -x_q^T K_q \Phi_q^{-1} x_q - v^T \Phi_q^{-1} x_q + k_{F_p} \left\|\bar{\theta}_p\right\| \left\|K_p \Phi_p^{-1} x_p\right\| \left\|x_q\right\| + \left\|\bar{\theta}_q\right\| \left\|F_q\right\| \left\|\Phi_q^{-1} x_q\right\|. \end{aligned} \tag{11}$$

Let v be designed as follows.

$$v = \begin{cases} \left(k_{F_q} \left\|\bar{\theta}_p\right\| \left\|\Phi_p^{-1} x_p\right\| \Phi_q + \left\|\bar{\theta}_q\right\| \left\|F_q\right\|\right)\left(x_q / \left\|x_q\right\|\right), & \text{if } \left\|x_q\right\| \ne 0, \\ 0, & \text{if } \left\|x_q\right\| = 0. \end{cases} \tag{12}$$

If $\left\|x_q\right\| \ne 0$, (11) becomes

$$\begin{aligned} \dot{V} &\leqslant - x_q^T K_q \Phi_q^{-1} x_q + k_F \left\|\bar{\theta}_p\right\| \left\|\Phi_p^{-1} x_p\right\| \left(x_q^T x_q\right) / \left\|x_q\right\| \\ &\quad + \left\|\bar{\theta}_q\right\| \left\|F_q\right\| \left(x_q^T \Phi_q^{-1} x_q\right) / \left\|x_q\right\| + k_F \left\|\bar{\theta}_p\right\| \left\|K_p \Phi_p^{-1} x_p\right\| \left\|x_q\right\| + \left\|\bar{\theta}_q\right\| \left\|F_q\right\| \left\|\Phi_q^{-1} x_q\right\| \\ &\leqslant -x_q^T K_q \Phi_q^{-1} x_q \\ &\leqslant 0, \end{aligned} \tag{13}$$

in the set that $x_p \in \Omega_p$ and $x_q \in \Omega_q$. If $\left\|x_q\right\| = 0$, $\dot{V}(t) \le 0$.

Theorem. Consider the system (1)-(2) under Assumptions 1-3. Let the control law be chosen as follows.

$$u=\begin{cases}-G_q^{-T}\left(\Phi_q G_p^T K_p \Phi_p^{-1} x_p + K_q x_q + (k_F \|\bar{\theta}_p\| \|\Phi_p^{-1} x_p\| \Phi_q + +\|\bar{\theta}_q\| \|F_q\|) \left(x_q / \|x_q\|\right)\right), & \text{if } \|x_q\| \neq 0, \\ -G_q^{-T}(\Phi_q G_p^T K_p \Phi_p^{-1} x_p + K_q x_q), & \text{if } \|x_q\| = 0.\end{cases} \tag{14}$$

Then, if $x_p(0) \in \Omega_p$ and $x_q(0) \in \Omega_q$, it follows that $x_p(t) \in \Omega_p$ and $x_q(t) \in \Omega_q$ for $t \in [0, \infty)$. The origin $x = 0$ is asymptotically stable.

Proof. We first show that $x_p(t) \in \Omega_p$ and $x_q(t) \in \Omega_q$ for $t \in [0, \infty)$. Since $\dot{V} \leq 0$ for $x_p(t) \in \Omega_p$ and $x_q(t) \in \Omega_q$, it is concluded that $V(x(t)) \leq V(x(0))$ for $t \in [0, \infty)$. According to the lemma, if $x_p(0) \in \Omega_p$ and $x_q(0) \in \Omega_q$, it follows that $x_p(t) \in \Omega_p$ and $x_q(t) \in \Omega_q$ for $t \in [0, \infty)$.

Now, we show that $x_p(t) \to 0$ and $x_q(t) \to 0$ as $t \to \infty$. If $\|x_q\| \neq 0$, the closed-loop system of (1)-(2) becomes

$$\dot{x}_p = F_p \theta_p + G_p x_q\,, \tag{15}$$

$$\dot{x}_q = F_q \theta_q + \Phi_q G_p^T K_p \Phi_p^{-1} x_p - K_q x_q - \left(k_F \|\bar{\theta}_p\| \|K_p \Phi_p^{-1} x_p\| \Phi_q + \|\bar{\theta}_q\| \|F_q\|\right)\left(x_q / \|x_q\|\right). \tag{16}$$

Let $S = \{ x_p(t) \in \Omega_p,\, x_q(t) \in \Omega_q : \dot{V}(x_p(t), x_q(t)) = 0\}$. Since $\dot{V} \leqslant -x_q^T K_q \Phi_q^{-1} x_q \leqslant 0$, $\dot{V} = 0$ implies that $x_q(t) = 0$. Then, $S = \{x_p(t) \in \Omega_p,\, x_q(t) \in \Omega_q : x_q(t) = 0\}$. Suppose that $x(t)$ is a trajectory that belongs identically to S, then $x_q(t) = 0$ implies that $\dot{x}_q(t) = 0$. According to Assumption 2, $\| F_p(x(t)) \| \leq k_{F_p} \|x_q(t)\|$, $x_q(t) = 0$ implies that $F_p(x(t)) = 0$. Then, from (15), $x_p(t) = c$, where c is a constant vector.

Since $x_q(t) = 0$, $\dot{x}_q(t) = 0$, and from the Assumption 2 $\| F_q(x(t)) \| \leq k_{F_q} \|x_q(t)\|$, (16) becomes $\Phi_q G_p^T K_p \Phi_p^{-1} c = 0$, and therefore $x_p(t) = c = 0$. The only solution that can stay identically in S is $x_p(t) = x_q(t) = 0$. Thus, it is obtained that x_p and x_q go to zero as $t \to \infty$. □

Remark 2. By using (14), the origin of (1)-(2) becomes the only equilibrium point for $x_p \in \Omega_p$, $x_q \in \Omega_q$. The linearization of (15)-(16) around the origin yields

$$\begin{bmatrix}\dot{x}_p \\ \dot{x}_q\end{bmatrix} = \begin{bmatrix}\dfrac{\partial F_p(x)\theta_p}{\partial x_p} & \dfrac{\partial F_p(x)\theta_p}{\partial x_q} + G_p(x) \\ \dfrac{\partial F_q(x)\theta_q}{\partial x_p} - \bar{x}_q^2 G_p^T(x) K_p \bar{x}_p^2 & \dfrac{\partial F_q(x)\theta_q}{\partial x_q} - K_q\end{bmatrix}_{x=0} \begin{bmatrix}x_p \\ x_q\end{bmatrix}. \tag{17}$$

Then, the spectrum of the matrix in (15) and (16) is completely assigned by K_p and K_q.

4. SIMULATION RESULTS

4.1. Simulation of Second Order Nonlinear System

In this subsection, to demonstrate the performance of the proposed control law, a numerical study of a single-input second order system is presented. The following nonlinear system is considered (for brevity, the independent variable *t* is omitted if there is no confusion).

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2 , \tag{18}$$

$$\dot{x}_2 = \theta_2 x_2^2 + x_1 + u , \tag{19}$$

where x_1 and x_2 are the states, which are constrained by $|x_1(t)| < \bar{x}_1$ and $|x_2(t)| < \bar{x}_2$, respectively, θ_1 and θ_2 are unknown parameters with $|\theta_1| < \bar{\theta}_1$ and $|\theta_2| < \bar{\theta}_2$, $\bar{\theta}_1$ and $\bar{\theta}_2$ are the bounds, respectively, and u is the control input. Let a Lyapunov function candidate for (18)-(19) be the following.

$$V = (1/2)\left(k_1 \ln\left(\Phi_1^{-1}\bar{x}_1^2\right) + \ln\left(\Phi_2^{-1}\bar{x}_2^2\right)\right), \tag{20}$$

where k_1 is a positive constant, $\Phi_1 = \bar{x}_1^2 - x_1^2$, and $\Phi_2 = \bar{x}_2^2 - x_2^2$. Let the control input be designed as

$$u = \begin{cases} -\left(k_1\Phi_1^{-1}\Phi_2 + 1\right)x_1 - k_2 x_2 - \left(k_1\Phi_1^{-1}\Phi_2\bar{\theta}_1 x_1^2 + \bar{\theta}_2 x_2^2\right)\operatorname{sgn}(x_2), & \text{if } x_2 \neq 0, \\ -\left(k_1\Phi_1^{-1}\Phi_2 + 1\right)x_1, & \text{if } x_2 = 0, \end{cases} \tag{21}$$

where k_2 is a positive constant. If $x_2 \neq 0$, the substitution of (21) into (18)-(19) yields:

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2 , \tag{22}$$

$$\dot{x}_2 = \theta_2 x_2^2 - k_1\Phi_1^{-1}\Phi_2 \; x_1 - k_2 x_2 - \left(k_1\Phi_1^{-1}\Phi_2\bar{\theta}_1 x_1^2 + \bar{\theta}_2 x_2^2\right)\operatorname{sgn}(x_2). \tag{23}$$

Using (22)-(23), the time derivative of V in (20) becomes

$$\begin{aligned} \dot{V} &= -k_2\,\Phi_2^{-1}x_2^2 - \left(k_1\Phi_1^{-1}\bar{\theta}_1 x_1^2 + \Phi_2^{-1}\bar{\theta}_2 x_2^2\right)|x_2| + k_1\Phi_1^{-1}\theta_1 x_1^2 \sin x_2 + \Phi_2^{-1}\theta_2 x_2^3 \\ &\le -k_2\,\Phi_2^{-1}x_2^2 \\ &\le 0, \end{aligned} \tag{24}$$

in the set that $|x_1(t)| < \bar{x}_1$ and $|x_2(t)| < \bar{x}_2$. From (21)-(23), it is concluded that, if $|x_1(0)| < \bar{x}_1$ and $|x_2(0)| < \bar{x}_2$, $|x_1(t)| < \bar{x}_1$ and $|x_2(t)| < \bar{x}_2$ for $t \in [0, \infty)$, and $x_1(t)$, $x_2(t) \to 0$ as $t \to \infty$.

A quadratic-function-based control law is also simulated. Let a Lyapunov function candidate for (18)-(19) be

$$V = (1/2)\left(k_1 x_1^2 + x_2^2\right), \tag{25}$$

and the control law be

$$u = \begin{cases} -\left(k_1 + 1\right)x_1 - k_2 x_2 - \left(k_1\bar{\theta}_1 x_1^2 + \bar{\theta}_2 x_2^2\right)\operatorname{sgn}(x_2), & \text{if } x_2 \neq 0, \\ -\left(k_1 + 1\right)x_1, & \text{if } x_2 = 0. \end{cases} \tag{26}$$

If $x_2 \neq 0$, the substitution of (26) into (18)-(19) yields:

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2 , \tag{27}$$

$$\dot{x}_2 = \theta_2 x_2^2 - k_1 x_1 - k_2 x_2 - \left(k_1\bar{\theta}_1 x_1^2 + \bar{\theta}_2 x_2^2\right)\operatorname{sgn}(x_2). \tag{28}$$

Using (27)-(28), the time derivative of the Lyapunov function V in (24) becomes

$$\begin{aligned}\dot{V} &= -k_2 x_2^2 - \left(k_1\bar{\theta}_1 x_1^2 + \bar{\theta}_2 x_2^2\right)|x_2| + k_1\theta_1 x_1^2 \sin x_2 + \theta_2 x_2^3 \\ &\le -k_2 x_2^2 \\ &\le 0.\end{aligned} \tag{29}$$

From (27)-(29), it is concluded that $x_1(t)$, $x_2(t) \to 0$ as $t \to \infty$.

A control law based on the receding horizon NMPC is also simulated, see [24] for details.

The performance of the control law based upon the barrier function, the second one based upon the quadratic function in, and the third one based upon the NMPC are compared. For the barrier-function-based and quadratic-function-based control law, the gains are set to $k_1 = 8$ and $k_2 = 4$, in which the eigenvalues of the closed-loop linearized system of (22)-(23) and (27)-(28) are set to $-2 \pm 2j$. The bounds of the states are $\bar{x}_1 = \bar{x}_2 = 0.4$. The unknown parameters θ_1 and θ_2 are normally distributed random signals with $|\theta_1|$, $|\theta_2| < 1$. The phase portraits of the system using the barrier-function-based, the quadratic-function-based, and the NMPC-based control laws are depicted in Figs. 1-3, respectively. The trajectories using the barrier-function-based control law (21) and the NMPC do not violate the constraints for all time when the initial conditions are $|x_1(0)|$, $|x_2(0)| < 0.4$. However, using the quadratic-function-based control law in (26), the trajectories violate their constraints when the initial conditions are close to the constraints.

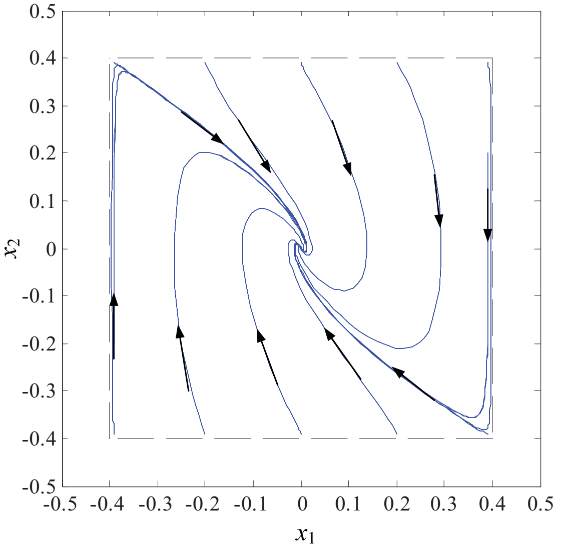

Fig. 1. Phase portrait of the system (18)-(19) using control law derived from the barrier function as the Lyapunov function.

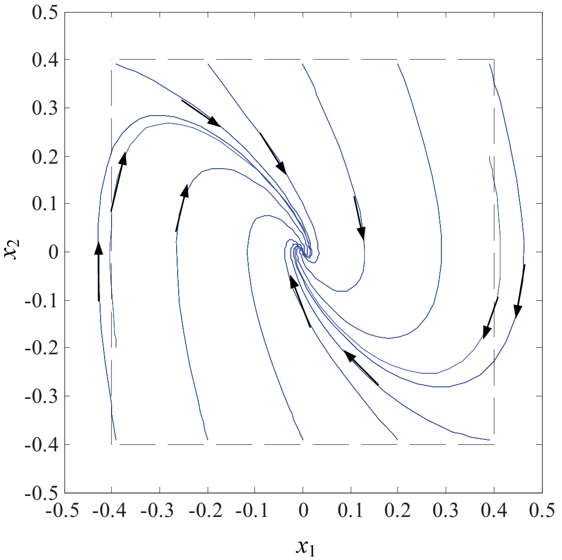

Fig. 2. Phase portrait of the system (18)-(19) using using the control law derived from the quadratic Lyapunov function.

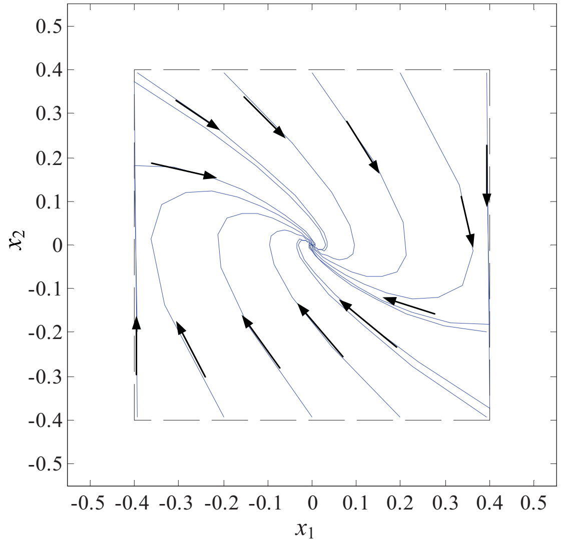

Fig. 3. Phase portrait of the system (18)-(19) using the NMPC.

Figs. 4-6 shows the motions of $x_1(t)$ and $x_2(t)$ with barrier-function-based, quadratic-function-based, and the NMPC control laws, respectively. The initial conditions are $x_1(0) = x_2(0) = 0.395$. Using the barrier-function-based control law, x_1 and x_2 reach zero in 3 s and do not violate the given constraints for all time. If using quadratic-function-based control law, x_1 violates the constraint from 0.01 s to 0.13 s, and x_2 violates from 0.17 s to 0.63 s, respectively. If using the NMPC, x_1 and x_2 go to zero in 10 s and their constraints are not violated for all time. It should be noted that the computation time of the control input of the barrier-function-based control law and the quadratic-function-based control law took 10^{-3} ms. On the other hand, the NMPC took 1.642 s for computing the control input. Fig. 7 depicts the control inputs of barrier-function-based, quadratic-function-based, and the NMPC control laws, respectively. The barrier-function-based control law provides the fast changing of the control input at the beginning compared to other control laws.

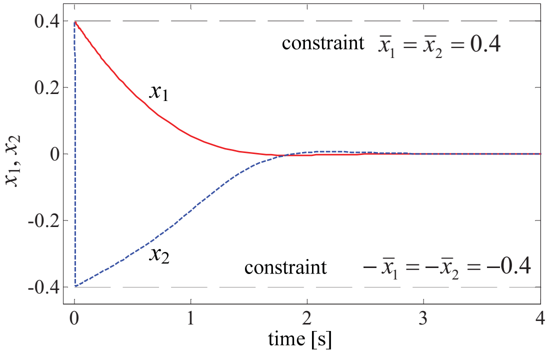

Fig. 4. $x_1(t)$ and $x_2(t)$ of system (18)-(19) using barrier-function-based control law: $x_1(t)$ and $x_2(t)$ converge to zero in 3 s while the constraints are not violated.

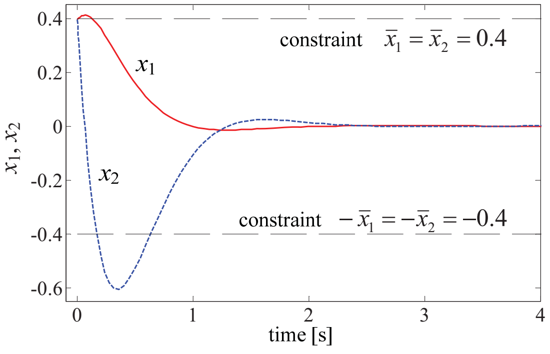

Fig. 5. $x_1(t)$ and $x_2(t)$ of system (18)-(19) using quadratic-function-based control law: $x_1(t)$ and $x_2(t)$ converge to zero in 3 s, but both violate the constraints.

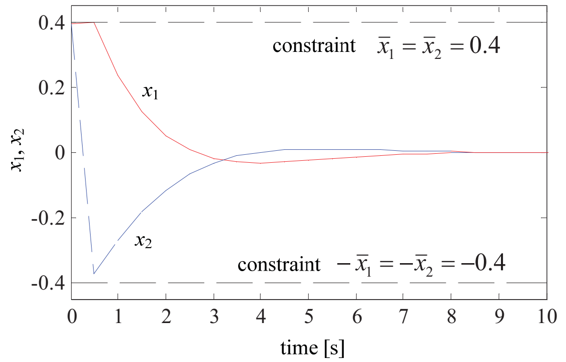

Fig. 6. $x_1(t)$ and $x_2(t)$ of system (18)-(19) using the NMPC control law: $x_1(t)$ and $x_2(t)$ converge to zero in 10 s while the constraints are not violated.

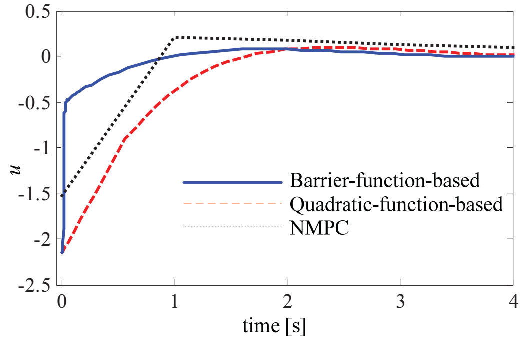

Fig. 7. Control input u used in Figs. 4-6.

4.2. Simulation of Multi-Input Nonlinear Systems

In this section, the following two-input nonlinear system is investigated.

$$\dot{x}_p = F_p\theta_1 + G_p x_q \,, \tag{30}$$

$$\dot{x}_q = F_q\theta_2 + G_q u \,, \tag{31}$$

where x_p and x_q are the states, $x_p = [x_1, x_2]^T$, $x_q = [x_3, x_4]^T$, that are constrained by $|x_i| < 1$, $i = 1\sim4$, θ_1 and θ_2 are the unknown parameters with $|\theta_1| < \bar{\theta}_1$ and $|\theta_2| < \bar{\theta}_2$, respectively, $F_p = [0.2\sin x_3,\ 0.2x_3x_4]^T$, $F_q = [-0.1x_2x_3,\ 0.2x_1x_4]^T$, ,

$$G_p = \begin{bmatrix} 1-\sin x_3 & 0.1 \\ 0.2 & 1-\sin x_1 \end{bmatrix}, \tag{32}$$

$$G_q = \begin{bmatrix} -1-x_1x_3 & 0.11x_3 \\ -0.1\sin x_4 & 1 \end{bmatrix}, \tag{33}$$

and $u = [u_1, u_2]^T$ are the inputs.

The control law based upon the barrier function is designed as follows:

$$u = \begin{cases} -G_q^{-T}\left(\Phi_q G_p^T K_p \Phi_p^{-1} x_p + K_q x_q + \left(k_F \bar{\theta}_1 \left\|K_p \Phi_p^{-1} x_p\right\| + \bar{\theta}_2 \left\|F_q\right\|\right)\left(x_q / \left\|x_q\right\|\right)\right), \text{if } \left\|x_q\right\| \neq 0, \\ -G_q^{-T}(\Phi_q G_p^T K_p \Phi_p^{-1} x_p), \qquad \text{if } \left\|x_q\right\| = 0, \end{cases} \tag{34}$$

where $K_p = K_q = 2I$, $k_F = 1$, $\Phi_p = \mathrm{diag}(\bar{x}_1^2 - x_1^2,\ \bar{x}_2^2 - x_2^2)$, $\Phi_q = \mathrm{diag}(\bar{x}_3^2 - x_3^2,\ \bar{x}_4^2 - x_4^2)$, and $\bar{x}_i$ = 1, i = 1~4. Fig. 8 shows the trajectories of xi(t), i = 1~4, of the system (30)-(31) when the control law (34) is used. The initial conditions are xi(0) = −0.995, i = 1~4. The unknown parameters are uniformly distributed random signals with bounds $\bar{\theta}_1 = 0.2$ and $\bar{\theta}_2 = 0.1$. All the trajectories converge to zero and do not violate the constraints at all time. Fig. 9 depicts the control inputs provided through (34).

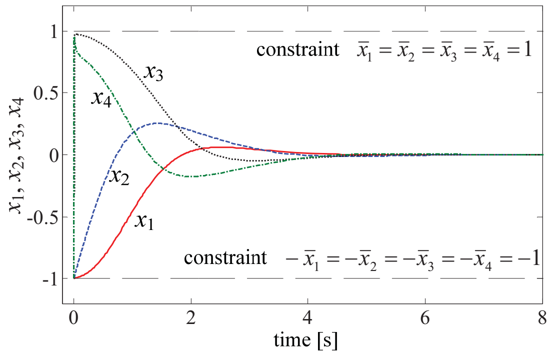

Fig. 8. $x_i(t)$, i = 1-4, of the system (30)-(31) with barrier-function-based control law.

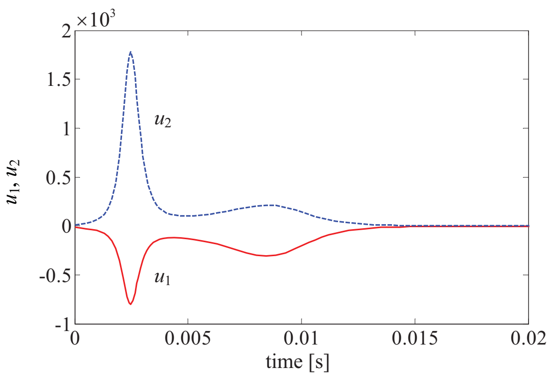

Fig. 9. Control input u_1 and u_2 used in Fig. 8.

5. CONCLUSIONS

The asymptotic stabilization control for nonlinear systems with state constraints was addressed. A barrier function was utilized as a Lyapunov function candidate in deriving the control law. The method in deriving the control law does not require offline optimization method, and thus the control algorithm is calculated very fast. The proposed control law guaranteed the asymptotic stability of the origin for all initial conditions inside the constraint region, while the given constraints were never violated. The desired spectrum of the linearized closed loop system was determined by designing the control gains, and thus a fast convergence of the states from all initial conditions within the constraint-region to the origin was achieved. The performance of the proposed control laws was compared with the one derived from a quadratic Lyapunov function and the NMPC. The proposed control law performed best in the aspects of convergence of the states to zero and in computing time of the control law among the three methods.

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